

12 - Predicate logic (continued)

Free vs bound variable. A variable is bound if it is within the scope of a quantifier; otherwise it is free.

$$[\forall x \in \mathbb{Z} : x^2 \geq x + y + z] \wedge [\exists y \in \mathbb{R} : (\forall z \in \mathbb{Z} : x^2 \geq y + z)]$$

Convention. In common speech, a free variable is implicitly universally quantified:

Example. If $x \geq 1$, then $x^2 \leq x^3$. \longrightarrow

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0 \quad \longrightarrow$$

Negating quantifiers. Intuitively \forall and \exists are supersized versions of \wedge and \vee :

So \neg distributes across terms and flips \wedge and \vee (de Morgan's law):

Quantifiers as loops.

\longrightarrow

```
for x in S:
  if not P(x) then
    return False
return True
```

\longrightarrow

```
for x in S:
  if Q(x) then
    return True
return False
```

Vacuous quantification.

Optional Homework due April 1 or 2.

Show your work. Answer without work receives no credit.

1. True or False?

- (a) $\exists x \in \mathbb{R} : x^2 > x$
- (b) $\exists x \in \mathbb{R} : x^2 = -1$
- (c) $\exists x \in \mathbb{R} : x^2 + 2 > 1$
- (d) $\forall x \in \mathbb{N} : (x^2 \neq x) \vee (x = 0) \vee (x = 1)$
- (e) $\exists n \in \mathbb{N} : n^2 \equiv 3 \pmod{4}$

2. Explain why this proposition is false: $\forall x \in \mathbb{R} : x^2 \geq x$

3. Let $C(x)$ ="x has a cat", $D(x)$ ="x has a dog", $F(x)$ ="x has a ferret.", S ={students in your class}. Formalize:

- (a) A student in your class has a cat, a dog, and a ferret.
- (b) All students in your class have a cat, a dog, or a ferret.
- (c) Some student in your class has a cat and a ferret but not a dog.
- (d) No student in this class has a cat, a dog, and a ferret.
- (e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals.

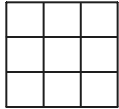
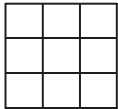
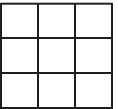
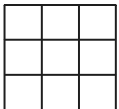
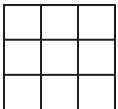
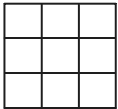
13 - Nested quantifiers (Liben-Nowell 3.5)

\forall commutes with \forall , \exists commutes with \exists , but \exists and \forall do not commute (so order matters):

$$\forall(x, y) \in S \times T : P(x, y) \equiv \forall x \in S : [\forall y \in T : P(x, y)] \equiv \forall y \in T : [\forall x \in S : P(x, y)]$$

$$\exists(x, y) \in S \times T : P(x, y) \equiv \exists x \in S : [\exists y \in T : P(x, y)] \equiv \exists y \in T : [\exists x \in S : P(x, y)]$$

Let $P(r, c) =$ "Row r column c is filled in the grid."

$\forall r \forall c P(r, c)$ $\equiv \forall c \forall r P(r, c)$	$\exists r \exists c P(r, c)$ $\equiv \exists c \exists r P(r, c)$	$\forall c \exists r P(r, c)$	$\forall r \exists c P(r, c)$	$\exists r \forall c P(r, c)$	$\exists c \forall r P(r, c)$
					

Negating nested quantifiers: "It is not the case that every grid cell is filled."

Translations:

1. A reciprocal of a real number x is another real number y such that $xy=1$.
 "Every nonzero real number has a reciprocal."

2. "There is a smallest natural number."

3. "There is no smallest positive real number."

4. "Every element in the sequence (x_1, x_2, x_3) appears more than once."

Optional Homework due April 8 or 9.

Show your work. Answer without work receives no credit.

1. True or False?

- (a) $\exists x \in \mathbb{R} : [\forall y \in \mathbb{R} : xy = 0]$
- (b) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : x = y^2]$
- (c) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : x^2 = y]$
- (d) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : (xy > 0) \vee (x = 0)]$
- (e) $\exists x \in \mathbb{R} : [\forall y \in \mathbb{R} : (xy > 0) \vee (y = 0)]$
- (f) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : (x \neq 0) \Rightarrow (xy > 0)]$

2. Formalize the statement that any two entries of the sequence (x_1, x_2, x_3) are distinct.

3. A sequence of real numbers $(x_1, x_2, x_3, x_4, \dots)$ is bounded if there exists an $M > 0$ such that $|x_i| \leq M$ for all $i \in \mathbb{N}$.

- (a) Use predicate logic to formalize the notion of a bounded sequence.
- (b) Negate the logical statement in (a) so that the resulting proposition does not use the symbol \neg .